Do Not Use the Arithmetic Mean to Average Multiples

By Gilbert E. Matthews, CFA,
Sutter Securities Inc. (San Francisco, Calif., USA)

Valuation professionals should not use the arithmetic mean of multiples. It is mathematically incorrect because it gives excessive weight to high multiples. A multiple is an inverted ratio with price in the numerator. Therefore, the harmonic mean should be used as the appropriate measure of central tendency. As a cross-check, the median should also be considered. Although abnormally low multiples can overly affect the harmonic mean, excluding outliers can correct this.[1]

The harmonic mean is calculated by: (i) taking the sum of the reciprocals of each value in a data series; (ii) dividing the sum by the number of values in the data series; and (iii) taking the reciprocal of that number. It is easy to calculate using Excel with the f_x [Insert Function] button or clicking on Fn+Shift+F3. Select HARMEAN, scroll over the datapoints to be averaged, and click. Alternatively, use the Σ [sum] function and replace SUM with HARMEAN in the formula bar.

The Median

The median is the midpoint of a range of numbers. It is a commonly used measure of central tendency approach and is a useful supplement to the harmonic mean for averaging multiples. In practice, the median of multiples is usually close to the harmonic mean.

The author has been using both the harmonic mean and the median in corporate valuations since the 1970s. In my experience, the median multiples are higher than the harmonic mean more often than they are lower than others; however, it is common for some medians of multiples within the same group of guideline companies to be lower than the harmonic mean while others are higher.

The median is not useful for small samples; with a limited number of guideline companies, the harmonic mean is the only useful measure of central tendency. The harmonic mean is superior to the median in another respect—because the median uses only one datapoint, it does not give any consideration to skewness in the data.

Support for the Harmonic Mean

Using the harmonic mean of multiples is not a new concept. Graham and Dodd's classic book, *Security Analysis,* used the harmonic mean to average P/E ratios in 1951. The use of the harmonic mean for averaging multiples was explained in detail in a book chapter on fairness opinions published in 1990.[2] A classic valuation book, Shannon Pratt's *Valuing a Business,*

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1   Outlying low multiples can distort the result. Since excluding a low outlier could be deemed to be selective, it is best to use a trimmed harmonic mean, excluding a high multiple for each low multiple excluded.

2   Gilbert E. Matthews and M. Mark Lee, "Fairness Opinions & Common Stock Valuations," in *The Library of Investment Banking*, Vol. 4, Robert L. Kuhn, ed. (Dow Jones Irwin, 1990): 381, 405-407.

explained in 1996, "The harmonic mean is used to give equal weight to each guideline company in summarizing ratios that have stock price or MVIC [market value of invested capital] in the numerator."[3]

Empirical analyses by Baker and Ruback in a 1999 Harvard working paper demonstrated that the arithmetic mean was a poor measure of central tendency for multiples of revenues, EBITDA and EBIT. They also concluded that the harmonic mean was somewhat better than the median.[4] Liu, Nissim, and Thomas, in a 2002 empirical study, arrived at the same conclusion.[5] Numerous subsequent studies have arrived at the same conclusion based on empirical data.[6] A few studies

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3   Shannon P. Pratt, Robert F. Reilly, and Robert P. Schweihs, *Valuing a Business: The Analysis and Appraisal of Closely Held Companies*, 3rd edition (New York: Irwin, 1996): 225.

4   Malcom Baker and Richard S. Ruback, "Estimating Industry Multiples." Working Paper. Harvard University (1999), pp. 4-5, available at hbs.edu/faculty/Publication%20Files/EstimatingIndustry_b4e64d71-c8fd-4a5e-b31a-623d3a7d02bc.pdf.

5   Jing Liu, Doron Nissim, and Jacob Thomas, "Equity Valuation Using Multiples." *Journal of Accounting Research* 40 (1) (2002): 135, 137, 157.

6   *E.g.,* Ingolf Dittmann and Ernst G. Maug, "Biases and Error Measures: How to Compare Valuation Methods," ERIM Report Series Reference No. ERS-2006-011-F&A; Mannheim Finance Working Paper No. 2006-07 (Aug. 25, 2008), pp. 2, 8, available at ssrn.com/abstract=947436; Stefan Henschke and Carsten Homburg, "Equity Valuation Using Multiples: Controlling for Differences Between Firms" (May 2009), p. 22, available at papers.ssrn.com/sol3/papers.cfm?abstract_id=1270812; Toby Tatum, "Harmonic Mean Value: The Appropriate Measure of Central Tendency," *Business Appraisal Practice* (3rd quarter 2011) 28, 31; Georgia Pazarzi, "Comparison of the Residual Income and the Pricing Multiples Equity Valuation Models," II *International Journal in Economics and Business Administration,* Issue 3, 88, 102 (2014); Jens Overgaard Knudsen, Simon Kold, and Thomas Plenborg, "Stick to the Fundamentals and Discover Your Peers," 73 *Financial Analysts Journal* 84, 104 (2017); William H. Black and Lari B. Masten, "Empirical Investigation of Alternative Measures of Central Tendency," 5 *Journal of Forensic Accounting Research* 216 (2020).

showed that the median was somewhat better than the harmonic mean,[7] others concluded that they were similar,[8] but all agreed that the arithmetic mean was an inferior approach.

A Flawed Contrary View

In contrast, a 2015 *BVU* article argues that "the harmonic mean should be avoided because it is inherently biased low."[9] The article's authors correctly conclude that the median is superior to the arithmetic mean, but they argue that the harmonic mean should not be used. They rely on unsound reasoning in rejecting the harmonic mean. Their conclusions would be valid if they were examining datapoints with price in the denominator, such as dividend yields. Their analyses fail because they do not take into account the basic reason why the harmonic mean is appropriate for multiples: the fact that a multiple is an inverted ratio because price is in the numerator.

The authors acknowledge that the arithmetic mean "is most frequently higher than the median,"[10] but they erroneously assert that the arithmetic mean gives the same weight to each multiple.[11] They fail to recognize that the reason

that the arithmetic mean is almost always higher is that it is upwardly biased by high multiples. Moreover, the distribution of multiples is almost always positively skewed.

They attempt to "prove" that the harmonic mean is biased using this statistically unsound analysis:

> To further test the bias of the harmonic mean, we used the random number generator in Excel. In Excel, if you type "rand()" into a cell, it will generate a random number between 0 and 1. Since the central tendency—median and average—of Excel's random generator *is* 0.50, we know in advance the true unbiased result.[12]

They then conclude that the mean and median in their test of "random numbers" were both 0.50 and the harmonic mean was 0.26. Based on this determination, they claim, "[W]e demonstrated that the harmonic mean is a biased low estimator of central tendency when data are distributed normally." They mistakenly assume that the appropriate measure of central tendency does not depend on the nature of the underlying data.

They concede the obvious fact (knowable even without using Excel) that the arithmetic mean and the median of numbers from 0.00 to 1.00 (or 0.01 to 0.99, or 0.10 to 0.90) is 0.50. If we take the reciprocals of numbers from 0.01 to 0.99 (analogous to ratios with price in the numerator), the harmonic mean and the median are 0.50 and the arithmetic mean is 0.19. This shows that, for reciprocals, it is the arithmetic mean that is a biased measure of central tendency. The question is when it is appropriate to average raw numbers or to average their reciprocals.

Is it more reasonable to average multiples using reciprocals? The exhibit shows that: (a) if an investor invested equal amounts in a $100,000 portfolio of four companies with P/E ratios of 50x, 25x, 15x, and 10x, the portfolio would have earnings of $5,667 and a multiple of 17.6x (the

7   *E.g.,* Volker Herrmann and Frank Richter, "Pricing with Performance-Controlled Multiples," 55 *Schmalenbach Business Review* 194, 212 (2003); Andreas Schreiner and Klaus Spremann, "Multiples and Their Valuation Accuracy in European Equity Markets," Working Paper (Aug. 13, 2007), p. 12, fn. 5, available at SSRN: ssrn.com/abstract=957352.

8   *E.g.,* Mingcherng Deng, Peter D. Easton, and Julian Yeo, "Another Look at Equity and Enterprise Valuation Based on Multiples" (April 2010), pp. 15-16, available at ssrn.com/abstract=1462794; Thomas Plenborg and Rene Coppe Pimentel, "Best Practices in Applying Multiples for Valuation Purposes," 59 *The Journal of Private Equity* 55, 59 (Summer 2016); Emanuel Bagnal and Enrico Cotta Ramusino, "Market Multiples and the Valuation of Cyclical Companies," 10 *International Business Research* (Issue 12) 246, 252 (2017).

9   Robert M. Dohmeyer, Herbert Kierulff, and Janae Castell, "Mean, Median, Harmonic Mean: Which Is Best?" *Business Valuation Update,* Jan. 2015, p. 1.

10 Id. at 4.

11 Id.

12 Id.

harmonic mean); and (b) if the investor bought equal amounts of earnings in each company (i.e., investing five times as much at a 50x multiple as in a 10x multiple), the portfolio's earnings would be $4,000 and its multiple would be 25x (the arithmetic mean).

Equal Investment vs. Equal Earnings

Equal investment:			Equal earnings:		
Invested	P/E	Earnings	Invested	P/E	Earnings
$25,000	50.0x	$500	$50,000	50.0x	$1,000
$25,000	25.0x	$1,000	$25,000	25.0x	$1,000
$25,000	15.0x	$1,667	$15,000	15.0x	$1,000
$25,000	10.0x	$2,500	$10,000	10.0x	$1,000
$100,000	17.6x	$5,667	$100,000	25.0x	$4,000

To give equal weight to each ratio with price in the numerator, it is necessary to use the harmonic mean—that is why it is the best measure of central tendency for multiples. The arithmetic mean gives five times as much weight to a 50x multiple compared to a 10x multiple, demonstrating that there is upward bias to an arithmetic mean of multiples. The harmonic mean of datapoints is always lower than the arithmetic mean. Because the arithmetic mean of multiples gives excessive weight to high multiples, it necessarily results in an overvaluation.

Weighted Harmonic Mean

Some valuation experts favor the use of a weighted harmonic mean.[13] This method is appropriate for calculating the multiple of a weighted index. However, use of a weighted mean requires a subjective judgment as to what factor to use for weighting the multiples. Several alternatives could be chosen, such as market capitalization, revenues, free cash flow, and net income. One study has concluded that the accuracy of the

harmonic mean can be improved by weighting the harmonic mean by the growth rate but not by other factors.[14]

Importantly, any weighting based on size necessarily gives more weight to larger guideline companies than to smaller ones. Why should larger companies be given greater weight? A weighted mean devalues the input from smaller guideline companies, even though the company being valued is commonly closer in size to them than to the larger ones.

Regression Analysis

Rather than using a measure of central tendency for multiples, some writers have used a regression analysis for valuation purposes.[15] Regression analyses are not useful unless there is a large number of observations.[16] This approach, like the weighted mean, brings in an element of subjective judgment: Which factors (e.g., revenues, profit margins, growth rate, market value, and payout ratio) should be considered in the regression analysis?

Some studies have found that the regression approach fails in empirical tests.

> Finally, a regression-based approach to directly estimating valuation multiples does

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13 E.g., Toby Tatum, "In Defense of Tatum's Law of Market Multiples," *Business Valuation Update,* April 2018, Special Supplement, p. 18.

14 Ian Cooper and Neophytos Lambertides, "Is There a Limit to the Accuracy of Equity Valuation Using Multiples?" (2014), p. 22, available at papers.ssrn .com/sol3/papers.cfm?abstract_id=2291869.

15 *See, e.g.* Mark Filler, "Letter to the Editor," *Business Valuation Update,* August. 2006, p. 20. For a discussion of the use of regression analysis (which is outside the scope of this article), *see, e.g.* Aswath Damodaran, *Investment Valuation,* 3rd edition (Wiley, 2012), pp..464-66, 562-69; Henschke and Homburg, pp. 3-18; Sanjeev Bhojraj, Charles Lee, and David Ng, "International Valuation Using Smart Multiples," working paper, Cornell University 2003, available at semanticscholar.org/paper/International-Valuation-Using-Smart-Multiples-Bhojraj-Lee/0e8ce3d2ffdc87fe 905a3b25bb9193f19326f2c6.

16 Baker and Ruback, p. 2.

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not necessarily improve valuation accuracy.... Further analysis reveals that the relationship between the financial ratios and ... multiples is nonlinear and hence, a linear regression model leads to suboptimal results.[17]

Despite Its Merits, the Harmonic Mean Is Not Widely Used

In 2016, Hitchner asked a group of valuation professionals which averages they typically use for multiples. The replies: 72% said the median, 31% said the arithmetic mean, and only 16% said the harmonic mean.[18]

Although average multiples are used in most fairness opinions, a review of fairness opinions on EDGAR shows that the median is used far more often than the arithmetic mean and that the harmonic mean of multiples is rarely used (other than in fairness opinions by Bear Stearns, which had used the harmonic mean since the 1970s).[19]

The harmonic mean for averaging multiples has rarely appeared in published court decisions, most likely because the expert witnesses did not discuss the subject. The author's Westlaw search found only five relevant cases. The harmonic mean was accepted twice and rejected thrice. A 1999 study that addressed Tax Court cases posited that arithmetic means were the poorest method for averaging multiples and that using reciprocals was the preferable method for averaging multiples. [20] However, the Tax Court has never discussed the harmonic mean.

Pratt wrote in 2001: "Although the harmonic mean is not used frequently, probably because it is unfamiliar to most readers of valuation reports, it is conceptually a very attractive alternative measure of central tendency."[21]

Despite numerous academic studies since then that demonstrate the harmonic mean's superiority, valuation professionals, investment bankers, or courts still do not widely use the harmonic mean. Many valuators are unfamiliar with the concept. Importantly, the harmonic mean is hardly ever discussed or even mentioned in books on corporate valuation.

Valuation practitioners should reject the use of the arithmetic mean for averaging multiples. Those who do not use the harmonic mean should review the available literature and decide whether they concur that it is the optimum measure of central tendency for ratios with price in the numerator.

Gilbert E. Matthews, CFA, is chairman emeritus and a senior managing director of Sutter Securities Inc. (San Francisco). He has more than 50 years of experience in investment banking and has spoken and written extensively on fairness opinions, corporate valuations, and litigation relating to valuations.

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17 Henschke and Homburg at 17. *See also, e.g.,* Volker Herrmann, *Marktpreisschätzung mit kontrollierten Multiplikatoren* (Cologne: Josef Eul Verlag, 2002): 233.

18 James R. Hitcher, "Poll Results Reflect Current Trends in Business Valuation," *Financial Valuation and Litigation Expert* (February-March 2017) at 6. Poll taken Feb. 3, 2016.

19 The author was chairman of Bear Stearns' Valuation Committee, which was responsible for all fairness opinions it issued from 1970 through 1995.

20 Randolph Beatty, Susan M. Riffe, and Rex Thompson "The Method of Comparables and Tax Court Valuations of Private Firms: An Empirical Investigation," 13 *Accounting Horizons* (Vol. 2) 177, 188-189 (1999).

21 Shannon P. Pratt, *The Market Approach to Valuing a Business* (New York; John Wiley & Sons, 2001): 133.